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LETTER TO THE EDITOR

On the Kac–Moody algebra of symmetries for a KdV equation in three dimensions

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Abstract. We have obtained the four different classes of symmetries for the equation

$$p_t+\partial_u^3 p+\partial_v^3 p-3\partial_u(p\partial_u\partial_v^{-1}p)-3\partial_v(p\partial_v\partial_u^{-1}p)=0$$

which is an analogue of the KdV equation in three spacetime dimensions. Of these four types of symmetries, two depend explicitly on the coordinates. It is then demonstrated that these generators of symmetries close on an infinite-dimensional Kac–Moody algebra.

One of the fundamental problems associated with a given nonlinear evolution equation (NEE) exactly solved by the inverse scattering transform is to construct the hierarchies of equations which have similar properties. The problem of finding an infinite number of Lie–Bäcklund symmetries is also related to it [1]. Exhaustive studies already exist for such properties of integrable two-dimensional systems [2]. Some properties of the three-dimensional KP equation are also well known [3]. But recent studies by Boiti *et al* [4] have produced many more integrable nonlinear systems in three dimensions. In this letter we follow the approach of admissible Lax operators to deduce the structure of an infinite class of symmetries for an equation which is a straightforward extension of the usual KdV equation to three dimensions. Four classes of symmetries are obtained of which two depend explicitly on the spacetime coordinates. Finally we show that these symmetries close on an infinite-dimensional Lie algebra.

The nonlinear equation under consideration can be written as

$$p_t+\partial_u^3 p+\partial_v^3-3\partial_u(p\partial_u\partial_v^{-1}p)-3\partial_v(p\partial_v\partial_u^{-1}p)=0. \tag{1}$$

It was demonstrated in [4] that the Lax operator associated with (1) is

$$L=\partial_u\partial_v-p \tag{2}$$

where the time evolution is given by

$$\psi_t=A\psi \tag{3}$$

A being an operator depending on (∂_u,∂_v) and also on (p, u and v). We call a class of operators A_k Lax admissible if

$$[L,A_k]=a_k \tag{4}$$

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where a_k is a multiplication operator only. An important point to note is that in our case $[L, A_k]$ is to be computed on the eigenspace of $L\psi = 0$. In the following we reproduce some such operators and their corresponding multiplication operator a_k:

$$
\begin{aligned}
&(a)\quad A_1 = \partial_u;\ a_1 = p_u \\
&(b)\quad B_1 = u\partial_u;\ b_1 = p + up_u \\
&(c)\quad C_1 = \partial_v;\ c_1 = p_v \\
&(d)\quad D_1 = v\partial_v;\ d_1 = p + vp_v
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
&(e)\quad A_3 = \partial_u^3 - 3(\partial_v^{-1}p_u)\partial_u \\
&\qquad a_3 = p_{3u} - 3p_u(\partial_v^{-1}p_u) - 3p(\partial_v^{-1}p_{2u}) \\
&(f)\quad B_3 = u\partial_u^3 + \partial_u^2 - \{3u\partial_v^{-1}p_u + \partial_v^{-1}p\}\partial_u \\
&\qquad b_3 = 2p_{2u} + up_{3u} - 4p(\partial_v^{-1}p_u) - 3up(\partial_v^{-1}p_{2u}) - 3up_u(\partial_v^{-1}p_u) - p_u(\partial_v^{-1}p) \\
&(g)\quad C_3 = \partial_v^3 - 3(\partial_u^{-1}p_v)\partial_v \\
&\qquad c_3 = p_{3v} - 3p_v(\partial_u^{-1}p_v) - 3p(\partial_u^{-1}p_{2v}) \\
&(h)\quad D_3 = v\partial_v^3 + \partial_v^2 - \{3v\partial_u^{-1}p_v + \partial_u^{-1}p\}\partial_v \\
&\qquad d_3 = 2p_2v + up_{3v} - 4p(\partial_u^{-1}p_v) - 3up(\partial_u^{-1}p_{2v}) - 3vp_v(\partial_u^{-1}p_v) - p_v(\partial_u^{-1}p) \\
&(i)\quad A_5 = \partial_u^5 - 5(\partial_v^{-1}p_u)\partial_u^3 - 5(\partial_v^{-1}p_2u)\partial_u^2 \\
&\qquad\qquad + [15\partial_v^{-1}\{p_u(\partial_v^{-1}p_u)\} - 5(\partial_v^{-1}p_{3u}) + 5\partial_v^{-1}\{p(\partial_v^{-1}p_{2u})\}]\partial_u \\
&\qquad a_5 = p_{5u} - 5p_{3u}(\partial_v^{-1}p_u) - 10p_{2u}\partial_v^{-1}p_{2u} - 10p_u\partial_v^{-1}p_{3u} + 5p_u \\
&\qquad\qquad \times \partial_v^{-1}\{p\partial_v^{-1}p_{2u}\} + 15p_u\partial_v^{-1}\{p_u\partial_v^{-1}p_u\} - 5p(\partial_v^{-1}p_{4u}) \\
&\qquad\qquad + 20p\partial_v^{-1}\{p_u\partial_v^{-1}p_{2u}\} + 5p\partial_v^{-1}\{p\partial_v^{-1}p_{3u}\} + 15\partial_v^{-1}\{p_{2u}\partial_v^{-1}p_u\}
\end{aligned}
\tag{6}
$$

and it is possible to go on and obtain higher-order operators. At this stage an important observation can be made. Let us define the commutators of two such operators A_i, B_j via the following rule [5]:

$$[\![A_i, B_j]\!] = A_i'[b_j] - B_j'[a_i] + [A_i, B_j] \tag{7}$$

where $A_i'[b_j]$ denote the Frechet derivative of A_i in the direction b_j:

$$A_i'(b_j] = \frac{\partial}{\partial\varepsilon} A_i[p + \varepsilon b_j]\big|_{\varepsilon=0} \tag{8}$$

and A_i, B_j denotes the usual commutator in the sense of vector fields. If we now compute $[\![A_3, B_3]\!]$ then the result turns out to be $3A_5$, whence

$$[\![A_3, B_3]\!] = 3A_5 \tag{9}$$

and we generate the higher-order operator A_5. Furthermore;

$$[\![A_5, B_3]\!] = 5A_7 \qquad \text{and so on.} \tag{10}$$

Therefore we call B_3 (as well as D_3) the generating Lax operators, and we have a recursive way of determining these higher-order operators. We note some simple commutation rules of such operators:

$$\begin{array}{lll} [\![A_1, B_3]\!] = A_3 & [\![A_3, B_1]\!] = 3A_3 & \\ [\![A_1, B_1]\!] = A_1 & [\![B_1, D_1]\!] = 0 & \\ [\![B_1, B_3]\!] = -3B_3 & [\![B_1, D_3]\!] = 0 & \\ [\![C_1, D_3]\!] = C_3 & [\![C_3, D_3]\!] = 3C_5 & \text{etc.} \end{array} \tag{11}$$

We note that equation (1) results due to the time evolution of ψ generated by $A_3 + C_3$.

Let us now suppose that p undergo a transformation $p \to p + \varepsilon\eta$, then η satisfies the linearized equation

$$\eta_t + \partial_u^3\eta + \partial_v^3\eta - 3\partial_u(\eta\partial_u\partial_v^{-1}p) - 3\partial_u(p\partial_u\partial_v^{-1}) - 3\partial_v(\eta\partial_v\partial_u^{-1}p) - 3\partial_v(p\partial_v\partial_u^{-1}) = 0. \tag{12}$$

Straightforward but laborious computation shows that the solutions of equation (12) are $\eta = a_i$, c_i and $\eta = ta_i + b_i$; $\eta = tc_i + d_i$, respectively the time-independent and time-dependent solution. So we have four classes of symmetry transformations for the three-dimensional KdV equation. Summarizing our above observations we can comment that we have constructed four classes of generators A_i, B_i, C_i and D_i, which generate the four classes of symmetries. These generators close on an infinite-dimensional Kac–Moody algebra, given by

$$\begin{array}{ll} [\![A_n, B_m]\!] = (n)A_{n+m-l} & [\![A_n, A_m]\!] = 0 \\ [\![C_n, D_m]\!] = nC_{n+m-l} & [\![C_n, C_m]\!] = 0 \\ [\![B_m, D_n]\!] = 0 & \\ [\![A_m, C_n]\!] = 0 & \\ [\![B_n, B_m]\!] = -mB_{n+m-1} & \\ [\![D_n, D_m]\!] = -mD_{n+m-1}. & \end{array}$$

So we have proved the existence of an infinite class of symmetries of 3D KdV problems, which is an important condition for the complete integrability of the system.

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